

2017 年第 34 屆成大數理比賽 - 數學

1. 關於 20^{17} 與 17^{20} 兩者的敘述，下列何者正確？ ($\log_{10} 2 = 0.3010$, $\log_{10} 17 = 1.2304$)

Which of the following statement about the numbers 20^{17} and 17^{20} is true? ($\log_{10} 2 = 0.3010$, $\log_{10} 17 = 1.2304$).

- (a) $20^{17} > 17^{20}$.
 - (b) 17^{20} 個位數字是 1. The unit digit of 17^{20} is 1.
 - (c) 20^{17} 的首位數字是 2. The first digit of 20^{17} is 2.
 - (d) 17^{20} 除以 3 的餘數為 2. The remainder of 17^{20} divided by 3 is 2.
 - (e) 以上皆非. None of above.
2. 由 2, 0, 1, 7 四個數字排列的四位數中，是 7 的倍數又是奇數的有幾個？
When we rearrange the numbers, 2, 0, 1, 7, how many four-digit odd numbers are a multiple of seven?
- (a) 0
 - (b) 1
 - (c) 2
 - (d) 3
 - (e) 以上皆非. None of above.

3. 把 $10!$ 以二進位表示為 $a_n \cdot 2^n + a_{n-1} \cdot 2^{n-1} + \cdots + a_1 \cdot 2 + a_0$, $a_0, a_1, \cdots, a_n \in \{0, 1\}$ ，下列敘述何者正確？

Express $10!$ as a binary number $a_n \cdot 2^n + a_{n-1} \cdot 2^{n-1} + \cdots + a_1 \cdot 2 + a_0$, $a_0, a_1, \cdots, a_n \in \{0, 1\}$, which of the following statement is true? ($\log_2 3 = 1.58$, $\log_2 5 = 2.32$, $\log_2 7 = 2.81$)

- (a) $a_0 = 1$.
 - (b) $a_1 = 1$.
 - (c) $a_n = 1$.
 - (d) $n = 22$.
 - (e) 以上皆非. None of above.
4. $y = f(x) = (x - 1)(x - 2) - 3x(x - 2) + 2x(x - 1)$,
 $f(2017) = ?$
- (a) 2016.
 - (b) 2017.
 - (c) 2018.
 - (d) 2019.
 - (e) 以上皆非. None of above.

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5. $f(x) = x^4 + 5x^3 + 10x^2 + 5x + 1 = a_4(x+1)^4 + a_3(x+1)^3 + a_2(x+1)^2 + a_1(x+1) + a_0$, 下列敘述何者正確? Which of the following statement is true for $f(x) = x^4 + 5x^3 + 10x^2 + 5x + 1 = a_4(x+1)^4 + a_3(x+1)^3 + a_2(x+1)^2 + a_1(x+1) + a_0$, where $a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}$.

- (a) $a_4, a_3, a_2, a_1, a_0 \in \mathbb{N}$.
- (b) $a_4 + a_3 + a_2 + a_1 + a_0 = 1$.
- (c) $a_0 = 1$.
- (d) $16a_4 + 8a_3 + 4a_2 + 2a_1 + a_0 = f(2)$.
- (e) 以上皆非. None of above

6. 關於方程式 $|x - 1| + |x - 3| = 2$ 的敘述何者正確?
Which of the following statement of equation $|x - 1| + |x - 3| = 2$ is true?

- (a) x 有非負整數解. x has non-negative solution.
- (b) x 有 2 正整數解. x has two positive integer solutions.
- (c) x 有無限多個正整數解. x has an infinite number of positive integer solutions.
- (d) x 有無限多的實數解. x has an infinite number of real solutions
- (e) 以上皆非. None of above.

7. $x^2 - 2x - 1 = 0$ 的兩個根為 α, β 且 $\alpha > \beta$. 令 $f(n) = \alpha^n + \beta^n$, 下列敘述何者錯誤?
The roots of $x^2 - 2x - 1 = 0$ are α and β , $\alpha > \beta$. Let $f(n) = \alpha^n + \beta^n$. Which of the following statement is false?

- (a) $\beta < 1$.
- (b) $f(0) = 2$.
- (c) $f(1) = 2$.
- (d) $f(1) = 2$.
- (e) $f(n+1) = 2f(n) + f(n-1)$

8. $\omega = \cos(\frac{2\pi}{2017}) + i\sin(\frac{2\pi}{2017})$, 下列敘述何者正確?
Which of the following statement is true for $\omega = \cos(\frac{2\pi}{2017}) + i\sin(\frac{2\pi}{2017})$?

- (a) $\sum_{k=0}^{2017} \omega^{2017k} = 1$.
- (b) $\sum_{k=0}^{2017} \omega^k = 1$.
- (c) $\sum_{k=1}^{2017} \omega^{2k} = 1$.
- (d) $(1 - \omega)(1 - \omega^2) \cdots (1 - \omega^{2017}) = 1$.
- (e) $(1 + \omega)(1 + \omega^2) \cdots (1 + \omega^{2017}) = 1$.

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9. 關於 $x^2 - 2y^2 = 1$ 的整數解，下列敘述何者正確？

Given $x^2 - 2y^2 = 1$, where $x, y \in \mathbb{Z}$, which of the following statement is true?

(a) 只有 $(3, 2), (3, -2), (-3, 2), (-3, -2)$ 四組解。

There are only four solutions: $(x, y) = (3, 2), (3, -2), (-3, 2), \text{and } (-3, -2)$.

(b) y 一定是偶數. y must be an even number.

(c) x 一定是 3 的倍數. x must be a multiple of three.

(d) $(x+y)(x-y)$ 不可能是平方數. $(x+y)(x-y)$ is not a square.

(e) 以上皆非. None of above.

10. $x > 0, y > 0$ 且 $x + y = 36$, 求 $\frac{1}{x} + \frac{4}{y}$ 的最小值.

Given $x > 0, y > 0$ and $x + y = 36$. Find the minimum of $\frac{1}{x} + \frac{4}{y}$.

(a) $\frac{25}{36}$

(b) $\frac{1}{9}$

(c) $\frac{5}{36}$

(d) $\frac{1}{4}$

(e) 以上皆非. None of above.

11. 全班 100 個學生，喜歡棒球者有 55 人，喜歡籃球有 73 人。設同時喜歡棒球且喜歡籃球者最多 a 人、最少 b 人，則

$a - b = ?$

There are 100 students in a class. 55 students like baseball and 73 students like basketball. Suppose that there are at most a students and at least b students who like both baseball and basketball. What is $a - b$?

(a) 18

(b) 27

(c) 28

(d) 45

(e) 以上皆非. None of above.

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12. 將 4 顆紅球，3 顆白球，1 顆黑球放進袋子中。假設拿出每個球的機率相同，且取後不放回，試問抽 3 次抽不到黑球的機率是多少？

There are four red balls, three white balls and one black ball in a bag. Suppose the probability of taking each ball is the same and we don't put the ball back after taking it out. What is the probability if we take out three balls and there is no black ball in the three balls?

(a) $\frac{42}{336}$

(b) $\frac{90}{336}$

(c) $\frac{210}{336}$

(d) $\frac{270}{336}$

(e) 以上皆非. None of above.

13. 有 20 張牌，牌上分別印上 1, 2, 3, 4, 5 五個數字，每張牌只有一個數字，且每個數字各有四張。每個人先發手上兩張牌，再在桌上攤開三張牌。每個人將手上的牌與桌上的牌湊成一個最大的五位數字，誰的五位數字大的話就誰贏。假設現在只有兩人對戰，Alice 跟 Bob。Alice 拿到 2 跟 4，桌上三張牌為 1, 3, 5。請問 Alice 贏的機率為？

There are 20 cards. Numbers 1, 2, 3, 4, 5 are printed on the cards. Each card has one number and each number has four cards. In a game, each player has two cards, and three cards are shown on the table. A player can make a maximum five-digit number by the cards in hand and on the table. The player who has the greatest number win the game. Suppose that the game has two players, Alice and Bob. Alice got 2 and 4. Three cards, 1, 3, 5, are on the table. What is the probability that Alice wins the game?

(a) $\frac{102}{210}$

(b) $\frac{51}{210}$

(c) $\frac{102}{380}$

(d) $\frac{51}{380}$

(e) 以上皆非. None of above.

14. $x \in \mathbb{R}, \cos(3x) = \frac{1}{2}$. $x = ?$

(a) $x = \pm(\frac{\pi}{9} + 2k\pi), k \in \mathbb{Z}$.

(b) $x = \pm(\frac{\pi}{3} + 2k\pi), k \in \mathbb{Z}$.

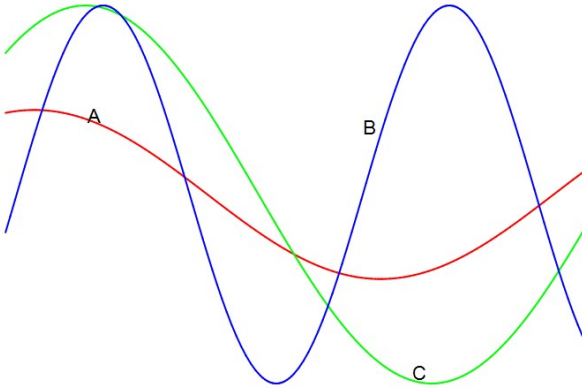
(c) $x = \pm(\frac{\pi + 2k\pi}{3}), k \in \mathbb{Z}$.

(d) $x = \pm(\frac{\pi}{9} + \frac{2k\pi}{3}), k \in \mathbb{Z}$.

(e) 以上皆非. None of above

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15. 如下簡單示意圖， $f(x) = \sin x + \cos x$, $g(x) = 3 \sin x + \cos x$, $h(x) = \sin(2x) - 3 \cos(2x)$ ，函數與圖形的配對為？ Please match the functions and their curves in the following illustration: $f(x) = \sin x + \cos x$, $g(x) = 3 \sin x + \cos x$, $h(x) = \sin(2x) - 3 \cos(2x)$.



- (a) $f:A$, $g:C$, $h:B$.
 (b) $f:C$, $g:A$, $h:B$.
 (c) $f:A$, $g:B$, $h:C$.
 (d) $f:C$, $g:B$, $h:A$.
 (e) 以上皆非. None of above.
16. $\triangle ABC$ 中一點 P , $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB} + \frac{2}{9}\overrightarrow{AC}$, 若延長 \overrightarrow{AP} 交 \overline{BC} 於 D 點，則下列敘述何者正確？
 A point P is in $\triangle ABC$, and $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB} + \frac{2}{9}\overrightarrow{AC}$. If the intersection of the extension of \overrightarrow{AP} and \overline{BC} is D , which of the following statement is true?

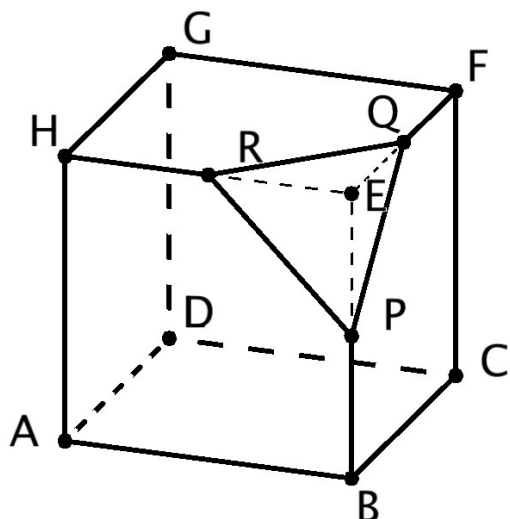
- (a) $\overline{BD}:\overline{CD} = 2:3$.
 (b) $\overline{BD}:\overline{CD} = 3:2$.
 (c) $\overline{AP}:\overline{PD} = 5:9$.
 (d) $\overline{AP}:\overline{PD} = 9:5$.
 (e) 以上皆非. None of above.

17. 平行四邊形 $ABCD$ ，其中 $A(0, 0)$, $B(a, 0)$, $C(a+b, c)$, $D(b, c)$ ，且 $a, b, c > 0$ 。而 L 為一斜率為 $m(m < 0)$ 的直線 $y = m(x - k)$ 。假設 $f(k)$ 為平行四邊形在 L 的下方 ($y < m(x - k)$) 的面積。下列敘述何者錯誤？
 A parallelogram $ABCD$ is located at $A(0, 0)$, $B(a, 0)$, $C(a+b, c)$, $D(b, c)$, where $a, b, c > 0$. L is a line $y = m(x - k)$ and the slope $m < 0$. Suppose that $f(k)$ is the area of parallelogram $ABCD$ and below L . Which of the following statement is false?

- (a) 可以找到 m 使得 $f(k)$ 在 $0 \leq k \leq a$ 時有可能是 k 的一次函數。 $f(k)$ may be a linear function of k when $0 \leq k \leq a$ for some slope m .
 (b) 可以找到 m 使得 $f(k)$ 在 $0 \leq k \leq a$ 時是 k 的二次函數。 $f(k)$ is a quadratic function of k when $0 \leq k \leq a$ for some slope m .
 (c) $f(k)$ 在 $a+b - \frac{c}{m} \leq k$ 時一定與 k 無關。 $f(k)$ is independent of k for $k \geq a+b - \frac{c}{m}$.
 (d) 如果 $a > b - \frac{c}{m}$ ，那麼 $f(k)$ 在 $b - \frac{c}{m} < k < a$ 時會是 k 的一次函數。 If $a > b - \frac{c}{m}$, $f(k)$ is a linear function when $b - \frac{c}{m} < k < a$.
 (e) 存在 k 使得 $f(k) = \frac{1}{2} \text{ABCD 面積}$ 。 There exists k such that $f(k)$ equals half of the area of $ABCD$.

18. 將一正方體截去一角，如下圖。其中 P, Q, R 為 \overline{BE} , \overline{EF} , \overline{EH} 中點。若 PQR 平面與 EFGH 平面的二面角為 θ ，求 $\sin \theta = ?$

We remove a corner from a cube, as indicated in the following figure. Three points P, Q, R are the mid-points of \overline{BE} , \overline{EF} , \overline{EH} , respectively. Let θ be the dihedral angle of plane PQR and plane EFGH. What is $\sin \theta$?



- (a) $\frac{\sqrt{3}}{3}$
 (b) $\frac{2}{3}$
 (c) $\frac{\sqrt{6}}{3}$
 (d) $\frac{2\sqrt{2}}{3}$
 (e) 以上皆非. None of above.
- 19.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{\sqrt[3]{x}} = ?$$

- (a) 0.
 (b) -1.
 (c) 1.
 (d) $\frac{2}{3}$
 (e) 以上皆非. None of above.

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20. 將 $y \leq x^n$, $0 \leq x \leq 1$ 與 $y \geq 0$ 所圍成的面積繞 x 軸旋轉一圈得到的體積為 $P(n)$, 繞 y 軸旋轉一圈得到的體積為 $Q(n)$ 。 $\lim_{n \rightarrow \infty} \frac{P(n)}{Q(n)} = ?$

Let $P(n)$ and $Q(n)$ are the volumes of rotating the area $\{y \leq x^n, 0 \leq x \leq 1, y \geq 0\}$ about x and y axes, respectively. What is $\lim_{n \rightarrow \infty} \frac{P(n)}{Q(n)} = ?$

- (a) 1.
- (b) $\frac{1}{2}$.
- (c) $\frac{2}{3}$.
- (d) $\frac{1}{4}$.
- (e) 以上皆非. None of above.

21. 在一個圓錐內接於一個半徑為 r 的球，它的頂點及底部的圓都在球上，請問這個圓錐最大的體積為何？
A cone is connected inside a sphere whose radius is r . The apex and the circle of the base of the cone are on the sphere. What is the maximum volume of the cone?

- (a) $\frac{16}{27} \pi r^3$
- (b) $\frac{16}{81} \pi r^3$
- (c) $\frac{32}{27} \pi r^3$
- (d) $\frac{32}{81} \pi r^3$

(e) 以上皆非. None of above.

22. A, B, C 是 2×2 矩陣，且 $AB = C$ ，其中 $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 7 & 3 \end{bmatrix}$ ，求 $B = ?$

A, B, C are 2×2 matrices, and $AB = C$. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 7 & 3 \end{bmatrix}$, what is $B = ?$

- (a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
- (b) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$
- (c) $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$
- (d) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

(e) 以上皆非. None of above.

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23. $x, a \in \mathbb{R}$. 「如果 $x^2 > a$, 那麼 $x > a$ 」這句話加上下面何種前提就一定正確的？
 Suppose $x, a \in \mathbb{R}$. Which of the following condition makes the statement “If $x^2 > a$, then $x > a$ ” always true?

- (a) a 是給定的比 1 大的正數. a is a given positive number and greater than 1.
- (b) x 是給定的比 1 小的正數. x is a given positive number and smaller than 1.
- (c) a 是給定的負數. a is a given negative number.
- (d) x 是給定的負數. x is a given negative number.
- (e) 以上皆非. None of above.

一數列 a_n 滿足下列遞迴式: $a_{n+1} = \begin{cases} \frac{a_n}{2}, & \text{如果 } a_n \text{ 是偶數} \\ 3a_n + 1, & \text{如果 } a_n \text{ 是奇數} \end{cases}$. 下列敘述何者錯誤？

A sequence a_n is generated by the following recursive formula: $a_{n+1} = \begin{cases} \frac{a_n}{2}, & \text{if } a_n \text{ is even;} \\ 3a_n + 1, & \text{if } a_n \text{ is odd.} \end{cases}$ Which of the following statements is false?

24.

- (a) $a_0 = 3, \Rightarrow a_7 = 1$
- (b) $a_0 = 5, \Rightarrow a_8 = 1$
- (c) $a_0 = 2^k, k \in \mathbb{N}, \Rightarrow a_{k+2} = 1$
- (d) $a_0 = 3 \cdot 2^k, k \in \mathbb{N}, \Rightarrow a_{k+7} = 1$
- (e) $a_0 = 5 \cdot 2^k, k \in \mathbb{N}, \Rightarrow a_{k+5} = 1$

25. 一數列依照以下的順序：1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, ...。以 0 分隔成多個區塊，而第 n 個區塊中有 n 個 1，則前 100 項的總和為？

A sequence is generated in the following: 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, The block which consists of 1 is separated by 0 and the number of 1 in n -th block is n . What is the sum of first 100 terms?

- (a) 66.
- (b) 77.
- (c) 78.
- (d) 88.
- (e) 以上皆非. None of above.